Dynamics of a Subsatellite System Supported by Two Tethers

A.K. Misra* and G.S. Diamond†
McGill University, Montreal, Canada

The dynamics of a subsatellite system connected to a main orbiter by two extensible but massless tethers is studied. The points of attachment of these two tethers are offset from the center of mass of the orbiter. Both constant-length and retrieval cases are considered. Lagrangian formulation for constrained systems is used to derive the nonlinear equations of motion. These along with the constraint equations are solved numerically. It is noted that the rotational motion in the plane of the attachment points can be damped quite effectively using a simple control scheme. Comparisons are made with the single-tether case. A two-tether system has superior rotational behavior, but seems to have disadvantages as far as longitudinal oscillations are concerned.

Introduction

TETHERS have the potential to play a significant role in future space operations. Bekey[1] has described how tethers may be used in atmospheric studies, power generation, artificial gravity facilities, construction of space platforms, orbital transfer, etc. A flight test of a tethered subsatellite system deployed from the Shuttle has been scheduled for 1987.

A large number of analyses have been made of the dynamics and control of tethered subsatellites deployed from the Shuttle using a single tether. These have been reviewed by Misra and Modi.[2] The objective of this paper is to present a preliminary investigation of the dynamics of a subsatellite supported by two tethers as opposed to one. The motivation behind this work came from the expectation that a two-tether system might have better dynamical characteristics compared with a one-tether system. In addition, it appears that two-tether systems (e.g., space cranes) will be applicable in space platform operations. To the authors' knowledge, this is the first analysis of a two-tether system.

Formulation of the Problem

The system under consideration is modeled as a point mass m connected to a much larger mass M that is revolving around the Earth in a circular orbit (see Fig. 1). The center of mass O of the system is assumed to coincide with that of the large body. A set of Cartesian coordinate axes is located at O such that the y_0 axis points away from the Earth along the local vertical, the z_0 axis is in the direction of forward motion of the main orbiter, and the x_0 axis completes the right-hand triad.

The subsatellite is held by two tethers that are attached to the main orbiter at points P_1 and P_2. The tethers are assumed to have negligible mass and to undergo uniform extensions. Thus, they have no transverse vibrations, but they do have longitudinal stretch. The position of the subsatellite may be specified by the distance L between its center of mass and that of the orbiter and by two rotations α (pitch) and γ (roll) in and out of the orbital plane, respectively. These rotations transform the orbital coordinate axes x_0, y_0, z_0 to a set of body coordinates axes x, y, z such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \tag{1}$$

The instantaneous lengths L_1 and L_2 of the two tethers are related to L, α, and γ through geometry as follows:

$$L_j = (L^2 + a_j^2 + b_j^2 + c_j^2 - 2a_jL\sin\gamma + 2b_jL\cos\alpha\cos\gamma + 2c_jL\sin\alpha\cos\gamma)^{1/2} \qquad j = 1,2 \tag{2}$$

where a_j, b_j, and c_j are coordinates of the point P_j in the $x_0 - y_0 - z_0$ frame. In addition, these lengths L_1 and L_2 are related to the corresponding instantaneous unstretched lengths L_{10} and L_{20} by the equation

$$L_j = L_{j0}(1+\epsilon_j), \qquad j = 1,2 \tag{3}$$

where ϵ_j is the strain in the jth tether. In certain control schemes, L_{j0} may be made to depend on the state variables. This is taken into account. In the following Lagrangian formulation, five generalized coordinates are used: α, γ, L, ϵ_1, and ϵ_2. It is clear that if the nominal lengths of the tethers at any instant are specified, the five generalized coordinates are not independent due to Eqs. (2) and (3). The constraint equations may be written as

$$(1+\epsilon_j)L_{j0} - (L^2 + a_j^2 + b_j^2 + c_j^2 - 2a_jL\sin\gamma + 2b_jL\cos\alpha\cos\gamma + 2c_jL\sin\alpha\cos\gamma)^{1/2} = 0 \tag{4a}$$

or in the abbreviated form

$$f_j(\epsilon_j, \alpha, \gamma, L) = 0, \qquad j = 1,2 \tag{4b}$$

In principle, one may solve for any two variables in terms of the other three and substitute in the energy expressions. Alternately, a Lagrangian formulation for constrained systems may be carried out using a set of Lagrangian multipliers. The latter approach leads to simple equations and is followed here.

The kinetic energy of the system may be written as

$$K = K_0 + \tfrac{1}{2}m\{\dot{L}^2 + L^2[\dot{\gamma}^2 + (\dot{\alpha}+\Omega)^2\cos^2\gamma]\} \tag{5}$$

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*Associate Professor, Department of Mechanical Engineering. Senior Member AIAA.

†Research Assistant, Department of Mechanical Engineering (presently at Pratt and Whitney, Longueuil, Quebec, Canada).

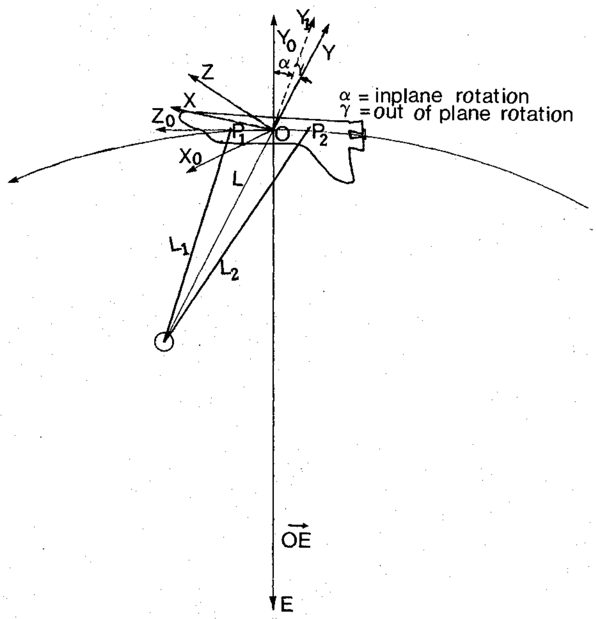

Fig. 1 Schematic diagram of the system.

where K_0 is the kinetic energy associated with the orbital motion and Ω the orbital angular velocity. Note that L and $\dot{L}$ have implicit contributions from the strains in the tethers. The potential energy V consists of the gravitational potential energy V_g and the strain energy V_s in the tether. Assuming a spherical Earth, V_g can be expressed as

$$V_g = V_{g0} + (\mu m L^2/2R_0^3)(1-3\cos^2\alpha\cos^2\gamma) \tag{6}$$

where V_{g0} is the part governing the orbital motion, μ the gravitational constant of the Earth, and R_0 the radius of the orbit. The potential energy due to longitudinal strain in the tethers is

$$V_s = \frac{1}{2}\sum_{j=1}^{2} A_j E_j L_{j0}\epsilon_j^2 U(\epsilon_j) \tag{7}$$

where A_j and E_j are the area of cross section and Young's modulus of the jth tether, respectively. Here, $U(\epsilon_j)$ is simply a unit step function, the use of which precludes any negative strain in the tether.

The portion of the tether wrapped around the tether reel, although elastic, is likely to have negligible elastic motion. This is due to the frictional force between the adjacent layers of the wrapped tether and that between the tether and the tether reel. As an example, 30 km of tether wrapped around a reel having a 0.5 m diameter involves approximately 20,000 turns. In that case, the amount of friction is likely to be too large to allow elastic motion of the wrapped tether. Thus, V_s receives no input from this portion of the tether.

The equations of motion can now be obtained from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial (V_g + V_s)}{\partial q_i} = Q_i + \sum_{j=1}^{2}\Lambda_j a_{ji} \tag{8}$$

where Λ_j are Lagrangian multipliers and q_i and Q_i the generalized coordinate and generalized force, respectively. a_{ji} are obtained from Eq. (4) through

$$a_{ji} = \frac{\partial f_j}{\partial q_i} \tag{9}$$

To nondimensionalize the equations of motion, a true anomaly θ is used as the independent variable and the lengths are divided by a reference length, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} = \Omega\frac{\mathrm{d}}{\mathrm{d}\theta} \qquad \ell = L/L_{\mathrm{ref}}$$
$$\ell_j = L_j/L_{\mathrm{ref}} \qquad \ell_{j0} = L_{j0}/L_{\mathrm{ref}} \tag{10}$$

Here, for convenience, the initial unstretched value of L is chosen as the reference length. With these, the equations for α, γ, and ℓ degrees of freedom are

$$\alpha'' + 2(\alpha'+1)\left[\left(\frac{\ell'}{\ell}\right) - \gamma'\tan\gamma\right] + 3\sin\alpha\cos\alpha$$
$$= \left(\frac{P_\alpha}{\ell^2\cos^2\gamma}\right) + \sum_{j=1}^{2}\left[\frac{\lambda_j(\tilde{b}_j\sin\alpha - \tilde{c}_j\cos\alpha)}{\ell\ell_j\cos\gamma}\right]$$
$$+ \sum_{j=1}^{2}\left[\frac{\lambda_j(1+\epsilon_j) - \frac{1}{2}\delta_j\epsilon_j^2 U(\epsilon_j)}{\ell^2\cos^2\gamma}\right]\frac{\partial \ell_{j0}}{\partial\alpha} \tag{11}$$

$$\gamma'' + 2\gamma'\left(\frac{\ell'}{\ell}\right) + [(\alpha'+1)^2 + 3\cos^2\alpha]\sin\gamma\cos\gamma$$
$$= \left(\frac{P_\gamma}{\ell^2}\right) + \sum_{j=1}^{2}\left[\frac{\lambda_j(\tilde{a}_j\cos\gamma + \tilde{b}_j\cos\alpha\sin\gamma + \tilde{c}_j\sin\alpha\sin\gamma)}{\ell\ell_j}\right]$$
$$+ \sum_{j=1}^{2}\left[\frac{\lambda_j(1+\epsilon_j) - \frac{1}{2}\delta_j\epsilon_j^2 U(\epsilon_j)}{\ell^2}\right]\frac{\partial \ell_{j0}}{\partial\gamma} \tag{12}$$

and

$$\ell'' + [1 - 3\cos^2\alpha\cos^2\gamma - (\alpha'+1)^2\cos^2\gamma - \gamma'^2]\ell$$
$$= P_\ell - \sum_{j=1}^{2}\left[\frac{\lambda_j(\ell - \tilde{a}_j\sin\gamma + \tilde{b}_j\cos\alpha\cos\gamma + \tilde{c}_j\sin\alpha\cos\gamma)}{\ell_j}\right]$$
$$+ \sum_{j=1}^{2}\left[\lambda_j(1+\epsilon_j) - \frac{1}{2}\delta_j\epsilon_j^2 U(\epsilon_j)\right]\frac{\partial \ell_{j0}}{\partial \ell} \tag{13}$$

where

$$P_\alpha = Q_\alpha/m\Omega^2 L_{\mathrm{ref}}^2, \quad P_\gamma = Q_\gamma/m\Omega^2 L_{\mathrm{ref}}^2, \quad P_\ell = Q_\ell/m\Omega^2 L_{\mathrm{ref}}$$
$$\lambda_j = \Lambda_j/m\Omega^2 L_{\mathrm{ref}}, \quad \delta_j = E_j A_j/m\Omega^2 L_{\mathrm{ref}}, \quad j=1,2$$
$$\tilde{a}_j = a_j/L_{\mathrm{ref}}, \quad \tilde{b}_j = b_j/L_{\mathrm{ref}}, \quad \tilde{c}_j = c_j/L_{\mathrm{ref}}, \quad j=1,2$$

and prime denotes differentiation with respect to θ. The equation for ϵ_i degrees of freedom may be written as

$$\delta_i\ell_{i0}\epsilon_i U(\epsilon_i) = P_{\epsilon i} + \lambda_i\ell_{i0} + \sum_{j=1}^{2}[\lambda_j(1+\epsilon_j)$$
$$-\tfrac{1}{2}\delta_j\epsilon_j^2 U(\epsilon_j)]\frac{\partial \ell_{j0}}{\partial\epsilon_i} \qquad i=1,2 \tag{14}$$

where $P_{\epsilon i} = Q_{\epsilon i}/m\Omega^2 L_{\mathrm{ref}}^2$.

One may note the absence of ϵ_i' and ϵ_i'' terms in Eq. (14). This is due to the form of the kinetic energy in Eq. (5) where the time derivatives of ϵ_i are implicit. The inertia terms associated with the strains appear implicitly through ℓ'' in Eq. (13). If the unstretched lengths L_{j0} are not functions of the generalized coordinates (for example, during uncontrolled dynamics), the second sum in each of Eqs. (11-13) and the sum in Eq. (14) drop out.

Numerical Integration

The seven variables in the formulation are the generalized coordinates α, γ, ℓ, ϵ_1, ϵ_2, and nondimensionalized Lagrangian multipliers λ_1 and λ_2. Three differential equations (11-13), two algebraic equations (14) for the epsilons, and the two constraint equations (4) are sufficient to determine these variables.

Solution of a system of mixed differential-algebraic equations (DAE) can often pose difficulties. If there are n generalized coordinates subjected to c constraints, then, in general, n second-order differential equations must be integrated (in the present case $n=5$, $c=2$). However, during this integration the constraints may be violated. In addition, the convergence can be quite slow. Gear[3] has shown that his stiff integration algorithm can be used to solve certain kinds of DAEs. Another approach, due to Baumgarte,[4] uses error feedback control to construct modified differential equations. Alternately, singular value decomposition techniques[5] can be used.

The set of DAEs here is somewhat simpler than the general case. First, there are only n-m (i.e., three) differential equations. In addition, Eq. (4) is linear in ϵ_j, while Eq. (14) is linear in λ_j. Thus, one can solve for ϵ_j using Eq. (4) to obtain

$$\epsilon_j=(\ell^2+\tilde{a}_j^2+\tilde{b}_j^2+\tilde{c}_j^2-2\tilde{a}_j\ell\sin\gamma+2\tilde{b}_j\ell\cos\alpha\cos\gamma$$
$$+2\tilde{c}_j\ell\sin\alpha\cos\gamma)^{1/2}/\ell_{j0}-1\equiv g_j(\alpha,\gamma,\ell) \tag{15}$$

Furthermore, Eq. (14) can be rewritten as

$$[A]\{\lambda\}=\{B\}$$

or

$$\{\lambda\}=[A]^{-1}\{B\} \tag{16}$$

where $[A]$ and $\{B\}$ are known functions of α, γ, ℓ, and ϵ_j, the latter being known functions of α, γ, and ℓ through Eq. (15). One can now use Eqs. (15) and (16) to convert, in principle, Eqs. (11-13) to depend on α, γ, and ℓ alone. The three differential equations can now be integrated numerically using standard procedures.

To start with, the second-order differential equations were converted into six first-order equations. During retrieval, using a length rate control law, two more first-order equations (for ℓ_{10} and ℓ_{20}) were added. These differential equations were solved using an IMSL (International Mathematical and Statistical Library) routine DGEAR based on Gear's method.

Results and Discussion

The tethers were assumed to be made of stainless steel (Young's modulus $=2\times10^{11}$ Pa) having a diameter of 0.5 mm. The subsatellite mass was chosen to be 150 kg. The altitude of the main orbiter was fixed at 220 km, corresponding to a mean orbital rate of 1.18×10^{-3} rad/s. For two-tether cases, the attachment points were assumed to be in the orbital plane and 10 m apart. One-tether cases were run by setting the offsets to zero and dividing the area of each tether by two. For all the runs, the initial values of α, γ, α', and γ' are 0, 0, 0.1, and 0.1, respectively.

At first, the motion of the subsatellite with constant-length tethers is examined. Two initial values of L_{j0} are considered: 1 km (short-tether case) and 100 km (long-tether case). For short lengths, there is very little difference between one- and two-tether cases as far as rotational motion is concerned. The pitch frequency slightly increases, while the maximum swing angle is reduced marginally for the two-tether case. The behavior of longitudinal tension, however, is markedly different. Consider the long-tether case first (Fig. 2). Apart from low-frequency variations in tension due to the rotational motion, there are significant high-frequency oscillations in the two-tether case

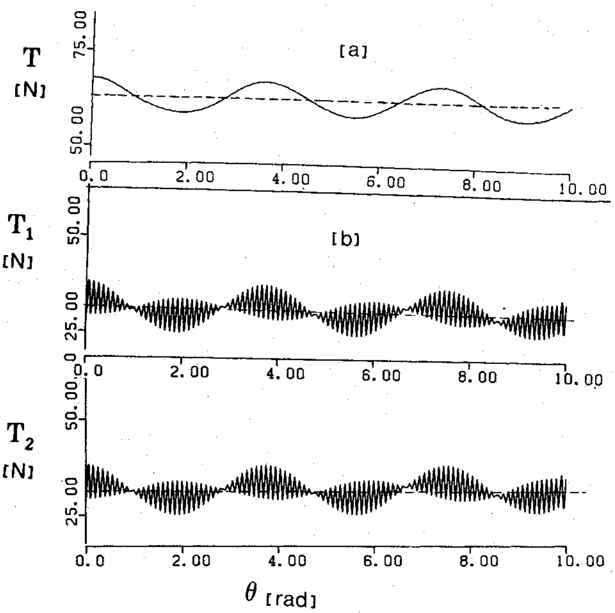

Fig. 2 Variation of tension (or longitudinal strain) with time for a) single-tether system and b) two-tether system ($L=100$ km).

associated with the elasticity of the tether. For the single-tether case, these elastic oscillations are too small to be visible in Fig. 2a. Apart from the high-frequency vibrations, the load is shared fairly evenly by the two tethers (Fig. 2b).

Both low- and high-frequency variations in tension are about a mean value of tension (dotted line) that can be calculated easily. Using a Newtonian approach, the steady-state tension in a vertical tether, arising due to the gravity gradient and orbital rotation, can be shown to be equal to $(3m+m_t)\Omega^2L$, where m and m_t are masses of the subsatellite and deployed part of the tether, respectively. (In the present study, the mass of the tether is neglected.) For the parameters chosen, this steady-state tension is approximately 63 N (approximately 31.5 N in each of the tethers in the two tether case).

For the short initial length, the tension behavior is quite interesting (see Fig. 3). Initially, the subsatellite is supported by the two tethers equally. As time progresses, all of the load is transferred onto one of the tethers, which vibrates with the natural frequency of one tether at that length. The number of times the load is transferred matches closely with twice the pitch frequency. One may also note the increase in amplitude of elastic vibrations every time this transfer occurs. The amplitude, however, remains constant while a given tether is under tension. This growth of vibration cannot be explained completely, but appears to be associated with the jerky transfer of load from one tether to another. The total mean tension is again given by $3m\Omega^2L$, i.e., 0.63 N for $L=1$ km. It may be pointed out that material damping in the tether has not been included in the dynamical model. This material damping would reduce the elastic oscillations; however, it appears that some sort of vibration control may be necessary to make the two-tether system feasible.

Dynamics of tethered systems during retrieval of the subsatellite is considered next. The retrieval is assumed to be exponential, i.e.,

$$L_{j0}'=-KL_{j0} \tag{17a}$$

or

$$L_{j0}=L_{j0}(0)\exp(-K\theta),\qquad j=1,2 \tag{17b}$$

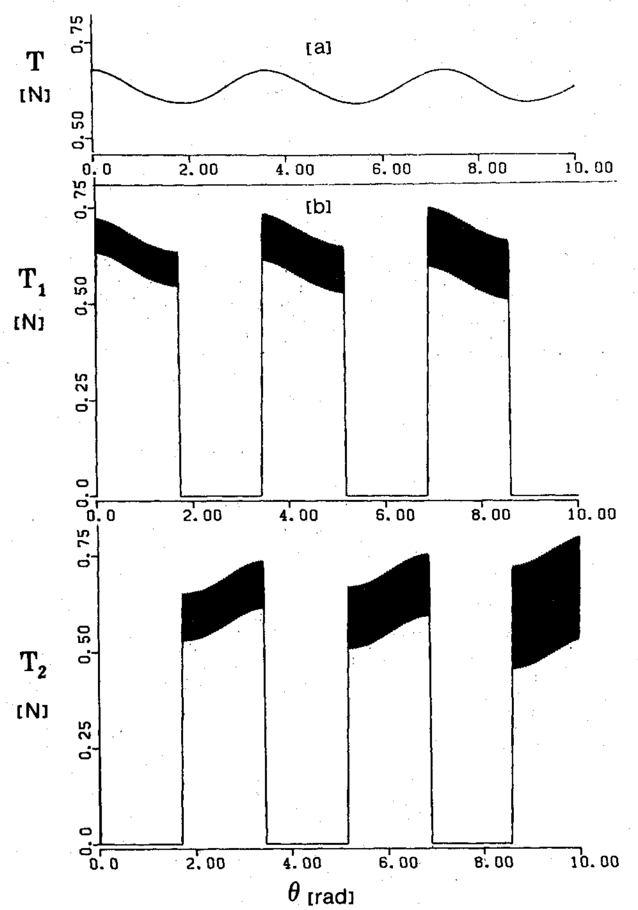

Fig. 3 Variation of tension (or longitudinal strain) with time for a) single-tether system and b) two-tether system ($L = 1$ km).

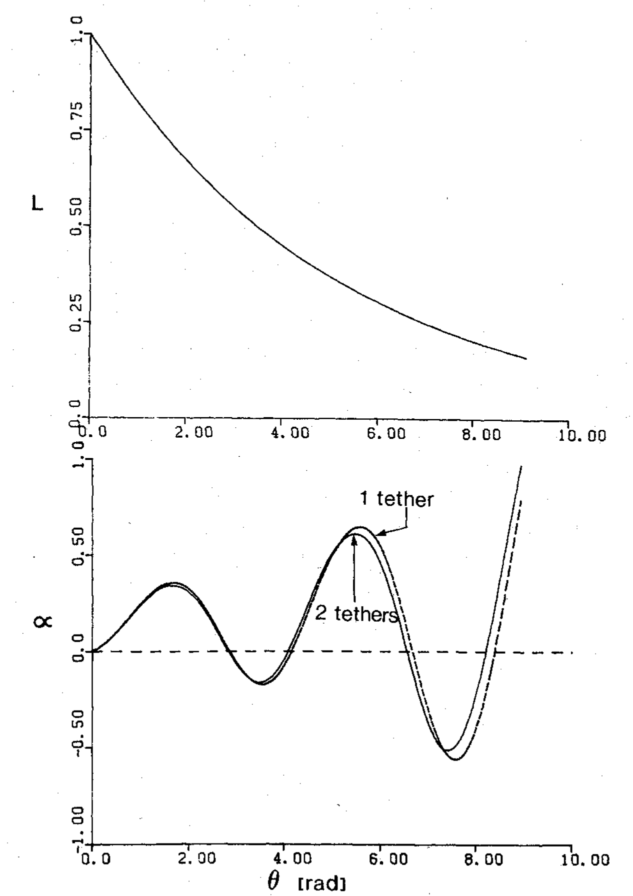

Fig. 4 Comparison of pitch behavior of single- and two-tether cases during exponential retrieval from 1 km ($K = 0.2$).

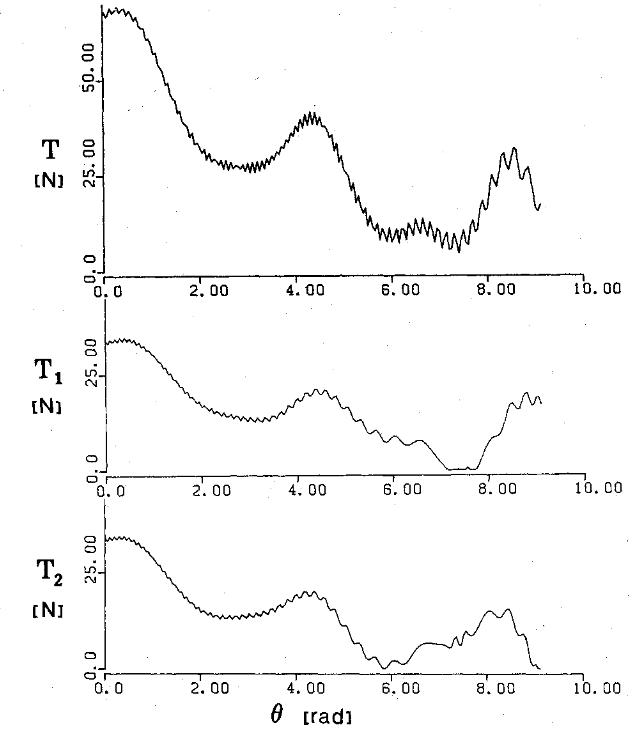

Fig. 5 Comparison of tension in single- and two-tether systems during exponential retrieval from 100 km ($K = 0.2$).

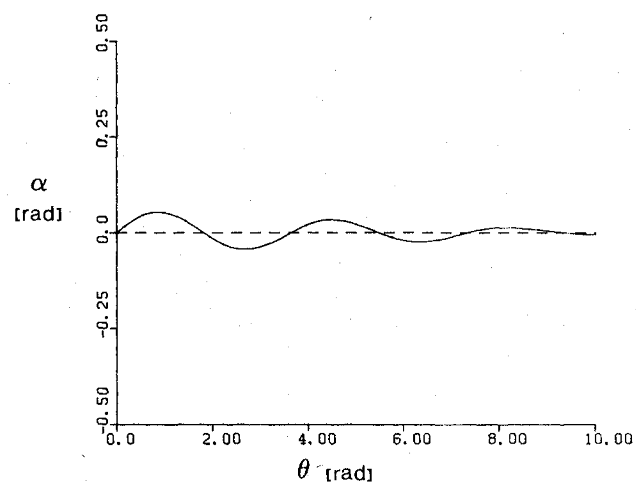

Fig. 6 Damping of pitch rotation using the two tethers ($L = 1$ km).

The pitch behavior for one- and two-tether systems during uncontrolled retrieval from 1 km using $K = 0.2$ is shown in Fig. 4. An increase in the pitch frequency as well as a reduction in the growth rate of the amplitude may be noted for the two-tether case. This implies that the two-tether system is relatively more stable as far as rotational motion is concerned.

Figure 5 compares the tension in the tethers for single- and two-tether cases during retrieveal from 100 km using $K = 0.2$. Initial behavior is similar except that for the single tether, as expected, the tension is twice that of the two-tether case. The load is shared more or less evenly by both the tethers in the two-tether cases during retrieval from 100 km using $K = 0.2$. differences appear in the tension in the two tethers. For very small lengths, each tether becomes alternately slack and taut as was observed in Fig. 3 (for constant-length short tethers).

An example of controlling the pitch motion using the two tethers is shown in Fig. 6. A simple on-off control is used as follows:

$$T_1 = T_e \text{ and } T_2 = 0, \text{ for } \alpha' > 0$$

$$T_1 = 0 \text{ and } T_2 = T_e, \text{ for } \alpha' < 0 \tag{18}$$

where T_j, $j = 1,2$, is the tension in the jth tether and T_e the sum of the equilibrium (steady-state) tensions in the two tethers. The roll motion can also be damped using a similar procedure if the offsets are not confined to the orbital plane.

Conclusions

Rotational behavior of a two-tether system appears to be superior to that of a single-tether system. During the stationkeeping phase, the rotational motion can be damped easily using a simple control scheme [Eq. (18)] involving the two tethers.

Two-tether systems may have poor performance as far as longitudinal vibrations are concerned, especially for short lengths. However, it might improve if structural damping is taken into account or in the presence of appropriate feedback control.

Further study using a more sophisticated model including the transverse vibrations is recommended to evaluate the effectiveness of the two-tether system in controlling the general dynamics during stationkeeping as well as retrieval phases.

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